Dihadron Production in Semi-inclusive Deep Inelastic Scattering: A Theoretical Overview

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Outline

I. Background
II. Lund/Artru and Gluon Radiation Models
III. Partial Wave Expansion of the Fragmentation Functions
IV. Cross Section
V. Spectator Model for Fragmentation
Background
Main Topics

**Spin**  Fundamental quantum number (more fundamental than mass). The group theory is identical to angular momentum.

**Proton**  Bound state of quarks and gluons, has spin 1/2 and mass 0.9 GeV

**Quark**  Fundamental particle, fermion (spin 1/2), interacts via “all” fundamental forces

**Gluon**  Fundamental particle, boson (spin 1), carries the strong nuclear force.
Standard Model of Particle Physics

Proton Models

- Early data suggested the proton was made of 2 u-quarks and 1 d-quark
- Pauli exclusion principle implies the spins of the u-quarks must be oppositely aligned
- The spin of the proton is then $\frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$
- Problem: data later showed that the quark masses equal only 10% of the proton mass.
  - Other 90% is binding energy, i.e. more quarks and gluons (called the sea)
  - The “original” 2 u and 1 d are called “valence quarks”
- How much do the quarks then contribute to the spin of the proton?
  - First measurements suggested 20-30%—The Spin Crisis!
Semi-Inclusive Deep Inelastic Scattering (SIDIS)

- Scattering: lepton interacting with proton
- Inelastic: produce additional particles
- Deep: highly off-shell virtual photon, probes internal structure of the proton
- Semi-Inclusive: the lepton and a few of the target fragments are measured
Experimental Access to Quark Spin

Deep Inelastic Scattering (DIS) \( e^\pm + p \rightarrow e^\pm + X \)

Inclusive DIS \( e^\pm + p \rightarrow h + X \) (\( h = \pi^\pm, \pi^0, K^\pm, \text{etc.} \))

Semi-Inclusive DIS (SIDIS) \( e^\pm + p \rightarrow e^\pm + h + X \)

More SIDIS \( e^\pm + p \rightarrow e^\pm + H + X \), with \( H \) a system of hadrons, e.g. \( \pi^+\pi^0 \) or \( K^+K^- \).

Inclusive pp \( p + p \rightarrow h + X \)

- Note: when colliding an electron or positron into a proton, it is not the electron that “hits” the proton, but rather a high energy photon
  - At HERMES, the cleanest data usually has the photon momentum between 30-90% of the lepton beam momentum.
  - The effective wavelength at HERMES was then between 50 to 150 am, while the other HERA experiment reached wavelengths near 1 am.

- When colliding two protons, it is possible for quarks, anti-quarks, and gluons from each of the protons to interact.
  - The results are more difficult to interpret, as several contributions of the above can contribute.
Cross Section Factorization

SIDIS cross section can be written

\[ \sigma^{ep \rightarrow ehX} = \sum_q DF \otimes \sigma^{eq \rightarrow eq} \otimes FF \]

Access integrals of DFs and FFs through Fourier moments of \( \phi_h, \phi_S, \phi_{R\perp} \) and Legendre polynomials in \( \cos \vartheta \).

Distribution and Fragmentation Functions

Distribution Functions (DF)

<table>
<thead>
<tr>
<th>nucleon</th>
<th>U</th>
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<tbody>
<tr>
<td>U</td>
<td>$f_1$</td>
<td></td>
<td>$h_{1T}^+$</td>
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<tr>
<td>L</td>
<td>$g_1$</td>
<td></td>
<td>$h_{1L}^+$</td>
</tr>
<tr>
<td>T</td>
<td>$f_{1T}^\perp$</td>
<td>$g_{1T}^\perp$</td>
<td>$h_1^\perp$</td>
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</table>

Unpol. | Pol.
-----|-----
$D_1$ | $H_1^\perp$

Fragmentation Functions (FF)

- Many more distributions at higher twist (an expansion in terms of the $Q^2$, the rest mass of the virtual photon)
- $f_1$ is the unpolarized distribution, $g_1$ the helicity distribution, and $h_1$ the transversity distribution.
- The $f_{1T}^\perp$ (the Sivers function) is related to orbital angular momentum of quarks.
- The pretzelosity function $h_{1T}^\perp$ is related to the shape of the proton.
- The Boer-Mulders function has polarized quarks in an unpolarized proton
- The “polarized” fragmentation function is known as the Collins function
Twist

Twist is rigorously defined as the difference between the order and the spin of an operator in an operator-product expansion.

In practice, twist describes the scaling with a relevant mass quantity divided by $Q$.

Leading twist is twist-2, i.e. an overall factor proportional to $1/Q^2$.

Higher twist also implies diagrams with more vertices and effects, even at leading order in $\alpha_s$. 
Optical Theorem

- Amplitudes of different \( |l', m'\rangle \) are summed before amplitude is squared.
- Analog two-dihadron amplitude includes sum the states of both dihadrons.
- Note: cross sections and physical quantities usually prefer direct-sum over direct-product bases.
  - E.g., physical meson states are basis elements \( |0, 0\rangle \) and \( |1, 0\rangle \), not basis elements \( |\frac{1}{2}, \frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle \).
  - New expansion: in terms of the \( |l, m\rangle \) state of the two Dihadron system.
Transverse Momentum

- In the $\gamma$-proton center-of-mass frame, the proton is moving with a large velocity.
- Initially, all effects from the motion of the quarks in directions transverse to the direction of the proton were considered completely negligible. (See Kane, et. al 1979)
- Thus the quarks are all assumed to be moving collinear with the proton (a ‘cold’ system)
- An inclusive asymmetry $A_N$ was found to be non-zero at several experiments at varying energies, with the only explanation being transverse momentum dependent (TMD) effects.
  - Two theories were suggested: one by D. Sivers with the TMD effect in the proton, and one by J. Collins with the TMD effect in the factorization process.
- Data taken 2002-2005 at HERMES fully demonstrate both of these transverse momentum effects (and others) at HERMES
  - Concurring results from other experiments are also now available.
The Lund/Artru and Gluon Radiation Models
Preliminaries

- Collinear SIDIS Dihadron cross section
  - Collinear access to transversity through two transverse target moments.
  - Transversity is coupled with “Collins-like” fragmentation functions $H_{1,OT}^{sp}$ and $H_{1,LT}^{pp}$, associated with $sp$ and $pp$ interference.

- TMD SIDIS Dihadron cross section
  - The Lund/Artru string fragmentation model predicts Collins function for pseudo-scalar and vector meson final states have opposite signs.

- Two types of fragmentation are usually defined
  - **Favored:** struck quark present in the observed particles.
  - **Disfavored:** struck quark not present in the observed particles.
Quark Spin and Meson Polarizations

- Mesons have one valence quark and one valence anti-quark
- The spins of the valence quark and anti-quark can be either aligned or anti-aligned
- One can either write the spins in the
  - Direct product basis: $|\frac{1}{2}, \pm \frac{1}{2}\rangle |\frac{1}{2}, \pm \frac{1}{2}\rangle$
  - Direct sum basis: $|1, m\rangle$ or $|0, 0\rangle$.
- One often writes $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$.
- In either case, there exists four basis elements
- The mass eigenstates are those of the direct sum basis
  - $|1, m\rangle$ represent three polarization of vector mesons
  - $|0, 0\rangle$ represent the one polarization of pseudo-scalar mesons
- For each pseudo-scalar meson, there exists a vector meson with identical quark content, only differing in the polarization of the quarks (up to mixing of mass flavor eigenstates)
Favored fragmentation modeled as the breaking of a gluon flux tube between the struck quark and the remnant.

Assume that the flux tube breaks into a $q\bar{q}$ pair with quantum numbers equal to the vacuum.

Expect mesons overlapping with $|\frac{1}{2}, \frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle$ states to prefer “quark left”.

- $|0, 0\rangle =$ pseudo-scalar mesons.
- $|1, 0\rangle =$ longitudinally polarized vector mesons.

Expect mesons overlapping with $|\frac{1}{2}, \frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle$ states to prefer “quark right”.

- $|1, \pm 1\rangle =$ transversely polarized vector mesons.

For the two $\rho_T$’s, “the Collins function” should have opposite sign to that for $\pi$.

For $\rho_L$, “the Collins function” is zero.
Favored fragmentation modeled as the breaking of a gluon flux tube between the struck quark and the remnant.

Assume that the flux tube breaks into a $q\bar{q}$ pair with quantum numbers equal to the vacuum.

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- $|0, 0\rangle = \text{pseudo-scalar mesons}$.
- $|1, 0\rangle = \text{longitudinally polarized vector mesons}$.

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- $|1, \pm 1\rangle = \text{transversely polarized vector mesons}$.

For the two $\rho_T$’s, “the Collins function” should have opposite sign to that for $\pi$.

For $\rho_L$, “the Collins function” is zero.
Gluon Radiation Fragmentation Model

- Disfavored frag. model: assume produced diquark forms the observed meson
- Assume additional final state interaction to set pseudo-scalar quantum numbers
- Assume no additional interactions in dihadron production.

- Exists common sub-diagram between this model and the Lund/Artru model.
- Keeping track of quark polarization states, sub-diagram for disfavored $|1, 1\rangle$ diquark production identical to sub-diagram for favored $|\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle$ diquark production.

- Implies that the disfavored Collins function for transverse vector mesons also has opposite sign as the favored pseudo-scalar Collins function
  - Thus fav. = disfav. for Vector Mesons
  - Data suggests fav. $\approx$ -disfav. for pseudo-scalar mesons.
Final result published in January

Significant $\pi^-$ asymmetry implies
$H_{1,\text{disf}} \approx -H_{1,\text{fav}}$

Pions have small, but non-zero asymmetry

Expect Collins moments negative for $\rho^\pm$.

Would like uncertainties on dihadron moments on the order of 0.02.
Partial Wave Analysis
Fragmentation Functions and Spin/Polarization

- Leading twist Fragmentation functions are related to number densities
  - Amplitudes squared rather than amplitudes
- Difficult to relate Artru/Lund prediction with published notation and cross section.
- Propose new convention for fragmentation functions
  - Functions entirely identified by the polarization states of the quarks, $\chi$ and $\chi'$
  - Any final-state polarization, i.e. $|\ell_1, m_1\rangle |\ell_2, m_2\rangle$, contained within partial wave expansion of fragmentation functions
- Exists exactly two fragmentation functions
  - $D_1$, the unpolarized fragmentation function ($\chi = \chi'$)
  - $H_1^\perp$, the polarized (Collins) fragmentation function ($\chi \neq \chi'$)
- New partial waves analysis proposed, using direct sum basis $|\ell, m\rangle$ rather than the direct product basis $|\ell_1, m_1\rangle |\ell_2, m_2\rangle$. 
Partial Wave Expansion

- Fragmentation functions expanded into partial waves in the direct sum basis according to

\[
D_1 = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} P_{\ell,m}(\cos \vartheta)e^{im(\phi_R-\phi_k)}D_{1|\ell,m}\rangle (z, M_h, |k_T|),
\]

\[
H_{1\perp} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} P_{\ell,m}(\cos \vartheta)e^{im(\phi_R-\phi_k)}H_{1\perp|\ell,m}\rangle (z, M_h, |k_T|),
\]

- Each term in pseudo-scalar and dihadron cross section uniquely related to a specific partial wave \(|\ell, m\rangle\).
- Cross section looks the same for all final states, excepting certain partial waves may or may not be present
  - Pseudo-scalar production is \(\ell = 0\) sector
  - Dihadron production is \(\ell = 0, 1, 2\) sector
  - Given the pseudo-scalar cross section (at any twist) can extrapolate cross section for other final states
Rigorous Definitions

- Fragmentation Correlation Matrix

\[
\Delta_{mn}(P_h, S_h; k) = \sum_X \int \frac{d^4 x}{(2\pi)^4} e^{i k \cdot x} \langle 0 | \Psi_m(x) | P_h, S_h; X \rangle \langle P_h, S_h; X | \overline{\Psi}_n(0) | 0 \rangle
\]

- Trace Notation

\[
\Delta^{[\Gamma]}(z, M_h, |k_T|, \cos \vartheta, \phi_R - \phi_k) = 4\pi \frac{z |R|}{16M_h} \int dk^+ \text{Tr} [\Gamma \Delta(k, P_h, R)] \bigg|_{k^- = P^-_h / z}.
\]

- Define fragmentation functions via trace relations

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<tr>
<th>FF</th>
<th>Previous Definitions</th>
<th>New Definition All Final States</th>
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<tr>
<td></td>
<td>Pseudo-Scalar</td>
<td>Dihadron</td>
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<tr>
<td>(D_1)</td>
<td>(\Delta^{[\gamma^-]})</td>
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<tr>
<td>(G_{1}^\perp)</td>
<td>– –</td>
<td>(\propto \Delta^{[\gamma^- \gamma^5]})</td>
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<td>(\Delta^{[(\sigma^1^-) \gamma^5]})</td>
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<tr>
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<td>– –</td>
<td>(\propto \Delta^{[(\sigma^2^-) \gamma^5]})</td>
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</table>
Relation with Previous Notation

- Real part of fragmentation function similar
- New definition of $D_1 \& H_1^\perp$
  - Adds “imaginary” part to $D_1 \& H_1^\perp$, instead of introducing new functions.
  - Functions are complex valued and depend on $Q^2$, $z$, $|k_T|$, $M_h$, $\cos \vartheta$, $(\phi_R - \phi_k)$.
- Comparing with similar trace definitions, e.g. PRD 67:094002, yields the relations

\[
D_1 \bigg|_{\text{Gliske}} = \left[ D_1 + i \frac{|R||k_T|}{M_h^2} \sin \vartheta \sin(\phi_R - \phi_k) G_1^\perp \right]_{\text{other}},
\]

\[
H_1^\perp \bigg|_{\text{Gliske}} = \left[ H_1^\perp + \frac{|R|}{|k_T|} \sin \vartheta e^{i(\phi_R - \phi_k)} \bar{H}_1^\times \right]_{\text{other}} = \frac{|R|^2}{|k_T|^2} H_1^\times \bigg|_{\text{other}},
\]

- Note: there are inconsistencies in the literature between definitions of $H_1^\times$, $\bar{H}_1^\times$, and $H_1'^\times$. 
Where is “the Collins function?”

- Consider direct sum vs. direct product basis

\[
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \left( \frac{1}{2} \otimes \frac{1}{2} \right) \otimes \left( \frac{1}{2} \otimes \frac{1}{2} \right),
\]

\[= (1 \oplus 0) \otimes (1 \oplus 0),
\]

\[= 2 \oplus 1 \oplus 1 \oplus 1 \oplus 0 \oplus 0.
\]

- Three \( \ell = 1 \) and two \( \ell = 0 \) cannot be separated experimentally
  - Theoretically distinguishable via Generalized Casimir Operators
- Longitudinal vector meson state \(|1, 0\rangle |1, 0\rangle\) is a mixture of \(|2, 0\rangle\) and \(|0, 0\rangle\)
  - Cannot access, due to \( \ell = 0 \) multiplicity
  - Model predictions for longitudinal vector mesons not testable
- Transverse vector meson states \(|1, \pm 1\rangle |1, \pm 1\rangle\) are exactly \(|2, \pm 2\rangle\)
  - Models predict dihadron \(H_{1}^{\perp}|2, \pm 2\rangle\) has opposite sign as pseudo-scalar \(H_{1}^{\perp}\).
  - Cross section has direct access to \(H_{1}^{\perp}|2, \pm 2\rangle\)
- Note: the usual IFF, related to \(H_{1}^{\perp}|1, 1\rangle\) is not pure \(sp\), but also includes \(pp\) interference.
Dihadron Twist-3 Cross Section

\[ d\sigma_{UU} = \frac{\alpha^2 M_h P_h \perp}{2\pi xyQ^2} \left( 1 + \frac{\gamma^2}{2x} \right) \]
\[ \times \sum_{\ell=0}^{2} \left\{ A(x, y) \sum_{m=0}^{\ell} \left[ P_{\ell,m} \cos(m(\phi_h - \phi_R)) \left( P_{\ell,m} \cos(m(\phi_h - \phi_R)) F_{UU,T} + \epsilon P_{\ell,m} \cos(m(\phi_h - \phi_R)) F_{UU,L} \right) \right] \right\} \]
\[ + B(x, y) \sum_{m=-\ell}^{\ell} P_{\ell,m} \cos((2 - m)\phi_h + m\phi_R) F_{UU} \]
\[ + V(x, y) \sum_{m=-\ell}^{\ell} P_{\ell,m} \cos((1 - m)\phi_h + m\phi_R) F_{UU} \]

\[ d\sigma_{UT} = \frac{\alpha^2 M_h P_h \perp}{2\pi xyQ^2} \left( 1 + \frac{\gamma^2}{2x} \right) |S_{\perp}| \sum_{\ell=0}^{2} \sum_{m=-\ell}^{\ell} \left\{ A(x, y) \left[ P_{\ell,m} \sin((m + 1)\phi_h - m\phi_R - \phi_S) \right] \right\} \]
\[ \times \left\{ P_{\ell,m} \sin((m + 1)\phi_h - m\phi_R - \phi_S) F_{UT,T} + \epsilon P_{\ell,m} \sin((m + 1)\phi_h - m\phi_R - \phi_S) F_{UT,L} \right\} \]
\[ + B(x, y) \left[ P_{\ell,m} \sin((1 - m)\phi_h + m\phi_R + \phi_S) F_{UT} \right] \]
\[ + P_{\ell,m} \sin((3 - m)\phi_h + m\phi_R - \phi_S) F_{UT} \]
\[ + V(x, y) \left[ P_{\ell,m} \sin(-m\phi_h + m\phi_R + \phi_S) F_{UT} \right] \]
\[ + P_{\ell,m} \sin((2 - m)\phi_h + m\phi_R - \phi_S) F_{UT} \]
\[ + P_{\ell,m} \sin((2 - m)\phi_h + m\phi_R - \phi_S) F_{UT} \]
Structure Functions, Unpolarized

\[
F_{UU,L}^{P \ell, m \cos(m \phi_h - m \phi_R)} = 0,
\]

\[
F_{UU,T}^{P \ell, m \cos(m \phi_h - m \phi_R)} = \begin{cases} 
\mathcal{J} \left[ f_1 D_1^{(\ell,0)} \right] & m = 0, \\
\mathcal{J} \left[ 2 \cos(m \phi_h - m \phi_k) f_1 \left( D_1^{(\ell,m)} + D_1^{(\ell,-m)} \right) \right] & m > 0,
\end{cases}
\]

\[
F_{UU}^{P \ell, m \cos((2-m) \phi_h + m \phi_R)} = -\mathcal{J} \left[ \frac{|p_T| |k_T|}{MM_h} \cos ((m - 2) \phi_h + \phi_p + (1 - m) \phi_k) \ h_1 H_{1}^{(\ell,m)} \right],
\]

\[
F_{UU}^{P \ell, m \cos((1-m) \phi_h + m \phi_R)} = -\frac{2M}{Q} \mathcal{J} \left[ \frac{|k_T|}{M_h} \cos((m - 1) \phi_h + (1 - m) \phi_k) \right]
\times \left( xhH_1^{(\ell,m)} + \frac{M_h}{M} f_1 \tilde{D}^{(\ell,m)} \right)
\]

\[
+ \frac{|p_T|}{M} \cos((m - 1) \phi_h + \phi_p - m \phi_k)
\times \left( xf^{(\ell,m)} + \frac{M}{M_h} h_1^{(\ell,m)} \tilde{H}^{(\ell,m)} \right).
\]

Can test Lund/Artru model with $F_{UU}^{\sin^2 \vartheta \cos(2 \phi_R)}$, $F_{UU}^{\sin^2 \vartheta \cos(4 \phi_h - 2 \phi_R)}$ via Boer-Mulder’s function.
Twist-2 Structure Functions, Transverse Target

\[ F_{UT,L}^{P_{\ell,m}} \sin((m+1)\phi_h - m\phi_R - \phi_S) = 0 \]

\[ F_{UT,T}^{P_{\ell,m}} \sin((m+1)\phi_h - m\phi_R - \phi_S) = -\mathcal{J} \left[ \frac{|p_T|}{M} \cos ((m + 1)\phi_h - \phi_p - m\phi_k) \right. \]

\[ \times \left( f_{1T}^{\perp} \left( D_1^{\ell,m} + D_1^{\ell,-m} \right) + \chi(m) g_{1T} \left( D_1^{\ell,m} - D_1^{\ell,-m} \right) \right) \bigg] , \]

\[ F_{UT}^{P_{\ell,m}} \sin((1-m)\phi_h + m\phi_R + \phi_S) = -\mathcal{J} \left[ \frac{|k_T|}{M_h} \cos ((m - 1)\phi_h - \phi_p - m\phi_k) h_1 H_1^{\perp |\ell,m} \right] , \]

\[ F_{UT}^{P_{\ell,m}} \sin((3-m)\phi_h + m\phi_R - \phi_S) = \mathcal{J} \left[ \frac{|p_T|^2 |k_T|}{M^2 M_h} \cos ((m - 3)\phi_h + 2\phi_p - (m - 1)\phi_k) h_1 H_1^{\perp |\ell,m} \right] . \]

- Can test Lund/Artru model with \( F_{UT}^{\sin^2 \vartheta \sin(\phi_h + 2\phi_R + \phi_S)} \) and \( F_{UT}^{\sin^2 \vartheta \sin(3\phi_h - 2\phi_R + \phi_S)} \) via transversity

- In theory, could also test Lund/Artru and gluon radiation models with \( F_{UT}^{\sin^2 \vartheta \sin(\phi_h + 2\phi_R - \phi_S)} \) and \( F_{UT}^{\sin^2 \vartheta \sin(5\phi_h - 2\phi_R - \phi_S)} \) via pretzelocity

- Data from SIDIS pseudo-scalar production indicate pretzelocity very small or possibly zero
Collinear Assumption and Structure Functions

- **TMD Structure function for the $|1, 1\rangle$ $A_{UT}$ moment**

\[
F_{UT}^{\sin \vartheta \sin (\phi_R + \phi_S)}(x, y, z, P_{h\perp}, p_T, k_T) = -j \left[ \frac{|k_T|}{M_h} \cos (\phi_p - \phi_k) h_1(x, p_T) H_1^{\perp|1,1\rangle}(z, zk_T) \right]
\]

- **Collinear assumption implies**

\[
\int d\phi_h \, dP_{h\perp} \, F_{UT}^{\sin \vartheta \sin (\phi_R + \phi_S)}(x, y, z, P_{h\perp}, p_T, k_T) \approx h_1(x) \, H_1^{\perp|1,1\rangle}(1)(z),
\]

with

\[
h_1(x) = \int dp_T \, h_1(x, p_T), \quad H_1^{\perp|1,1\rangle}(1)(z) = \int dk_T \frac{|k_T|}{M_h} H_1^{\perp|1,1\rangle}(z, zk_T).
\]
Collinear versus TMD Moments

- It is not the particulars of the DF or FF that make a moment survive in the collinear case, but rather the \( \sum m = 0 \) (necessary condition).
  - Moments with \( h_1^\perp H_1^\perp|\ell,m\rangle \) (Boer-Mulders moments)
    - \( h_1^\perp \) has \( \chi \neq \chi' \), and thus \( \Delta m = -1 \)
    - \( H_1^\perp \) similarly has \( \Delta m = -1 \).
    - Final state polarization must have \( m = 2 \) in order that \( \sum m = 0 \).
    - Only surviving moment in collinear dihadron production is \( |2,2\rangle \).
  - Moments with \( h_1 H_1^\perp|\ell,m\rangle \) (Collins moments)
    - \( h_1 \) has \( \Delta m = 0 \).
    - \( H_1^\perp \) again has \( \Delta m = -1 \).
    - Collinear moments are \( |1,1\rangle, |2,1\rangle \).
- Can also look for the \( m \) which cancels the \( \phi_h \) dependence

\[
F_{UU}^{P,\ell,m} \cos((2-m)\phi_h+m\phi_R) = -J \left[ \frac{|p_T| |k_T|}{MM_h} \cos ((m - 2)\phi_h + \phi_p + (1 - m)\phi_k) h_1^\perp H_1^\perp|\ell,m\rangle \right],
\]

\[
F_{UT}^{P,\ell,m} \sin((1-m)\phi_h+m\phi_R+\phi_S) = -J \left[ \frac{|k_T|}{M_h} \cos ((m - 1)\phi_h - \phi_p - m\phi_k) h_1 H_1^\perp|\ell,m\rangle \right],
\]
Spectator Model of Dihadron Fragmentation
Collinear Dihadron Spectator Model

- Exists only one model for polarized dihadron fragmentation functions
  - Focuses on collinear fragmentation
- The model is a spectator model
  - Optical theorem used to compute the scattering amplitude of $p\gamma^*\bar{p}'\gamma'^* \rightarrow H\bar{H}'$.
  - A single particle “spectator” is assumed to mediate between $p\gamma H$ and $\bar{p}\gamma \bar{H}$ vertices.
  - Spectator forced to be on-shell, with mass $M_s \propto M_h$.
- Model assumes single spectator for both hadron pairs and vector mesons.
  - This causes the amplitudes to be summed, rather than the cross sections
- The leading twist fragmentation correlation matrix is computed from the tree level diagram.
- Integration over transverse momenta is performed before extracting fragmentation functions via trace relationships.
TMD Dihadron Spectator Model

- One can use the same correlator to extract TMD fragmentation functions
  - Just do not integrate over transverse momentum.
  - Convenient to apply new partial wave analysis after Dirac trace algebra.
  - Numeric studies show need for additional $k_T$ cut-off.

- Original model intended for $\pi^+\pi^-$ pairs
  - Adding flavor dependence allows generalization to $\pi^+\pi^0$, $\pi^-\pi^0$ pairs.
  - Slight change to vertex function allows generalization to $K^+K^-$ pairs.
  - Slight change to vertex function and allows generalization to $K^+K^-$ pairs.

- Unfortunately, the model only includes partial waves of the Collins function for $\ell < 2$.
  - Instead, one can set $|2, \pm 2\rangle$ partial waves proportional to either $H_{1\perp}^{\ell,m}$ for $\ell \leq 1$ or to $D_1^{\ell,m}$ for $\ell \leq 2$. 
The tree-level diagram implies the following fragmentation correlation function

\[ \Delta^q(k, P_h, R) = \left\{ \begin{array}{l}
|F^s|^2 e^{-\frac{k^2}{\Lambda_s^2}} k (k - P_h + M_s) \frac{\delta}{k}
+ |F^p|^2 e^{-\frac{k^2}{\Lambda_p^2}} k R (k - P_h + M_s) \frac{\delta R}{k}
+ F^s F^p e^{-\frac{k^2}{\Lambda_{sp}^2}} k R (k - P_h + M_s) \frac{\delta R}{k}
+ F^s F^p e^{-\frac{k^2}{\Lambda_{sp}^2}} k F (k - P_h + M_s) \frac{\delta F}{k}
\end{array} \right. \]

\[ \times \frac{1}{(2\pi)^3} \frac{1}{k^4} \delta \left( (k - P_h)^2 - M_s^2 \right) e^{-\frac{k^2}{\Lambda_b^2}}. \]

The cut-offs are imposed by assuming certain vertex functions.

Fragmentation functions can be obtained by applying trace-definitions.
Results of the Model Calculation

\[
\frac{16\pi^2 M_h k^4}{|R|} \left| D_1^{0,0} \right\rangle = \left( \frac{z^2 |k_T|^2 + M_s^2}{1 - z} \right) \left[ |F^s|^2 e^{-2 \frac{k^2}{\Lambda_s^2}} - R^2 |F^p|^2 e^{-2 \frac{k^2}{\Lambda_p^2}} \right]
\]

\[
\frac{16\pi^2 M_h k^4}{|R|} \left| D_1^{1,1} \right\rangle = -2M_s |R| |k_T| \left[ \text{Re} \left( F^s F^p \right) e^{-2 \frac{k^2}{\Lambda_s^2}} \right]
\]

\[
\frac{16\pi^2 M_h k^4}{|R|} \left| D_1^{1,0} \right\rangle = -2 \frac{M_s |R|}{z M_h} \left( M_h^2 + z^2 |k_T|^2 \right) \left[ \text{Re} \left( F^s F^p \right) e^{-2 \frac{k^2}{\Lambda_s^2}} \right]
\]

\[
\frac{16\pi^2 M_h k^4}{|R|} \left| D_1^{2,2} \right\rangle = |k_T|^2 |R|^2 \left[ |F^p|^2 e^{-2 \frac{k^2}{\Lambda_p^2}} \right],
\]

\[
\frac{16\pi^2 M_h k^4}{|R|} \left| D_1^{2,1} \right\rangle = \frac{|k_T||R|^2}{z M_h} \left( M_h^2 + z^2 |k_T|^2 + \frac{1}{2} z^2 k^2 \right) \left[ |F^p|^2 e^{-2 \frac{k^2}{\Lambda_p^2}} \right],
\]

\[
\frac{16\pi^2 M_h k^4}{|R|} \left| D_1^{2,0} \right\rangle = \left( \frac{|R|^2}{z^2 M_h^2} \left( M_h^2 + z^2 |k_T|^2 \right) \left( M_h^2 + z^2 |k_T|^2 + z^2 k^2 \right) \left[ |F^p|^2 e^{-2 \frac{k^2}{\Lambda_p^2}} \right],
\]

\[
D_1^{\ell, -m} = D_1^{\ell, m}.
\]
Model Calculation for Fragmentation Functions

\[
\frac{8\pi^2 k^4}{|R|} H_1^\perp |1,1\rangle = -\frac{|R|}{|k_T|} \left( k^2 + |k_T|^2 \right) \left( \left( 1 - z^2 \right) k^2 - z^2 |k_T|^2 \right) \\
\times \left[ \text{Im} \left( F_s^* F^p \right) e^{-2 \frac{k^2}{\Lambda^2_{sp}}} \right],
\]

\[
\frac{8\pi^2 k^4}{|R|} H_1^\perp |1,0\rangle = \frac{1}{z} M_h |R| \left( z k^2 - 2 \left( M_h^2 + z^2 (k^2 + |k_T|^2) \right) \right) \\
\times \left[ \text{Im} \left( F_s^* F^p \right) e^{-2 \frac{k^2}{\Lambda^2_{sp}}} \right],
\]

\[
\frac{8\pi^2 k^4}{|R|} H_1^\perp |1,-1\rangle = -M_h^2 |R||k_T| \left[ \text{Im} \left( F_s^* F^p \right) e^{-2 \frac{k^2}{\Lambda^2_{sp}}} \right].
\]

▶ Note again the absence of the \( H_1^\perp |2,m\rangle \) partial waves.
Conclusions and Summary
Conclusions and Summary

- The Lund/Artru and (new) gluon radiation model
  - Can verify the predictions regarding the signs of certain structure functions
- New partial wave analysis
  - Increases understanding and aids in interpretation
  - Simplifies notation
  - Allows computation of the sub-leading twist cross section
- TMD Spectator Model for Dihadron Fragmentation
  - Only available model for TMD polarized dihadron production
  - Unfortunately, predicts $|2, \pm 2\rangle$ states to be zero.